Find the solution of the one-dimensional wave equation

$$u_{tt} = u_{xx}$$

under the following conditions

1) $u(0,x) = u_0 \sin \pi x$
2) $u_t(0,x) = 0$
3) $u(t,0) = u(t,1) = 0$

Interpretation: A string of length *l=1* is fixed at *x=0* and *x=1* with initial sin-shaped elongation without any initial velocity.

Solution:

$$u = u(t,x) = T(t)X(x)$$
$$\frac{d^2T}{dt^2}X = \frac{d^2X}{dx^2}T$$
$$\frac{d^2T}{dt^2}\frac{1}{T} = \frac{d^2X}{dx^2}\frac{1}{X} = \alpha$$

$$\frac{d^2T}{dt^2} - \alpha T = 0$$
$$\lambda^2 - \alpha = 0$$

$$\frac{d^2X}{dx^2} - \alpha X = 0$$
$$\lambda^2 - \alpha = 0$$

$\alpha > 0:$ $T(t) = c_1 e^{\sqrt{\alpha}t} + c_2 e^{-\sqrt{\alpha}t}$
$\alpha = 0:$ $T(t) = c_1 + c_2 t$
$\alpha < 0:$ $T(t) = c_1 \cos\sqrt{-\alpha}t + c_2 \sin\sqrt{-\alpha}t$

$\alpha > 0:$ $X(x) = a_1 e^{\sqrt{\alpha}x} + a_2 e^{-\sqrt{\alpha}x}$
$\alpha = 0:$ $X(x) = a_1 + a_2 x$
$\alpha < 0:$ $X(x) = a_1 \cos\sqrt{-\alpha}x + a_2 \sin\sqrt{-\alpha}x$

$CaseI : \alpha > 0:$ $u(x,t) = (c_1 e^{\sqrt{\alpha}t} + c_2 e^{-\sqrt{\alpha}t})(a_1 e^{\sqrt{\alpha}x} + a_2 e^{-\sqrt{\alpha}x})$
$CaseII : \alpha = 0:$ $u(x,t) = (c_1 + c_2 t)(a_1 + a_2 x)$
$CaseIII : \alpha < 0:$ $u(x,t) = (c_1 \cos\sqrt{-\alpha}t + c_2 \sin\sqrt{-\alpha}t)(a_1 \cos\sqrt{-\alpha}x + a_2 \sin\sqrt{-\alpha}x)$

Impose conditions:

Case I, II: Fulfill condition 3) only if $a_1 = a_2 = 0$; trivial solutions
Case III: set $-\alpha = \beta^2$, then using condition 3):
$u(x,0) = (c_1 \cos\beta t + c_2 \sin\beta t)(a_1 \cos 0) = 0; a_1 = 0$
$u(x,1) = (c_1 \cos\beta t + c_2 \sin\beta t)(a_2 \sin\beta) = 0, \beta = k\pi$

There are infinitely many solutions of the form:

$$u_k(x,t) = (c_{1k}\cos k\pi t + c_{2k}k\sin k\pi t)\sin k\pi x$$

and hence:

$$u(x,t) = \sum_k u_k = \sum_k (c_{1k}\cos k\pi t + c_{2k}k\sin k\pi t)\sin k\pi x$$

Use condition 1):

$$u(0,x) = \sum_k (c_{1k}\cos k\pi t + c_{2k}\sin k\pi t)\sin k\pi x = u_0\sin \pi x; \qquad for \quad k=1, c_{11}=u_0; c_{1k}=0(k\geq 2)$$

Use condition 2):

$$u_t(t,x) = \sum_k (-k\pi c_{1k}\sin k\pi t + k\pi c_{2k}\cos k\pi t)\sin k\pi x;$$

$$u_t(0,x) = \sum_k (k\pi c_{2k}\cos k\pi t)\sin k\pi x = 0; \qquad c_{2k}=0 \qquad for \quad all \quad k$$

And hence, the solution to the problem has been shown to be:

$$u_t(t,x) = u_0(\cos \pi t)\sin \pi$$